The Bayesian Cramér-Rao lower bound in Astrometry

Studying the impact of prior information
In the location of an object

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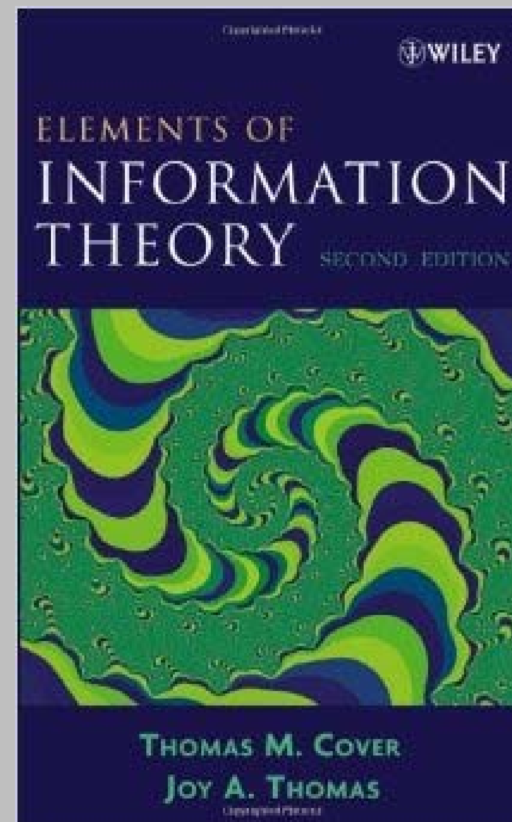
To appear on Astronomy & Astrophysics

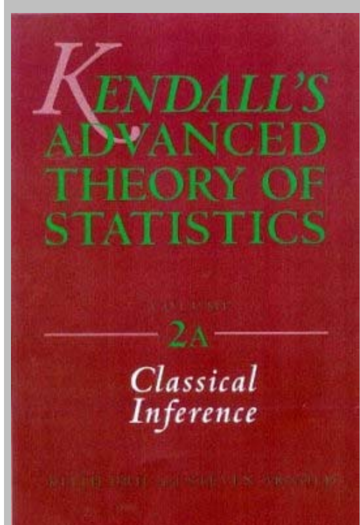
Given a set of measurements I_i (*i*=1…*n*) that follow and underlying distribution f_θ that depends on parameter θ:

What is the *minimum variance* of the parameter ?
How do we estimate the parameter ?



Optimal parameter estimation can, in general, be formulated using "decision-making" theory (Cover & Thomas 2006).
Claude Shannon, 1948: "A mathematical theory of communication" (transmission of information over a noisy channel).

This framework can also provide absolute lower bounds to the *uncertainty* of these estimators.



The Cramer-Rao (CR) lower uncertainty bound determines the minimum theoretical variance achievable by *any* unbiased estimator (Stuart, Ord, & Arnold 2004).

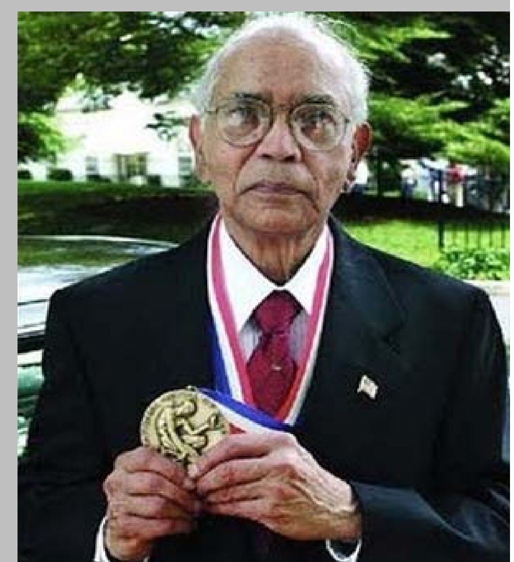
Clasical parameter estimation & the Cramér-Rao minimum variance bound

$$\hat{\theta}() \equiv \arg\min_{\hat{\theta}} \mathrm{Var}(\hat{\theta}(I_1, \ldots, I_n))$$

$$= \arg\min_{\hat{\theta}} \mathbb{E}_{I_1,\ldots,I_n \sim f_\theta^n} (\hat{\theta}(I_1, \ldots, I_n) - \theta)^2$$

Untractable, as it requires knowledge of the unknown θ $\Rightarrow$ performance bounds

Calyampudi Radakrishna Rao (1945)
"Information and accuracy attainable in the estimation of statistical parameters"
Bulletin of the Calcutta Mathematical Society **37**: 81–89



Harald Cramér (1946)
“A contribution to the theory of statistical estimation”
Scandinavian Actuarial Journal **1**: 85-94

Let $\hat{\theta}()$ be an unbiased estimator of θ. If we define the function $L(I_1, ..., I_n; \theta)$ as the likelihood of the observation given the model parameter θ, and we have n independent random variables I_i driven by the probability function f_θ, then the Cramér-Rao bound states that:

$$Var(\hat{\theta}(I_1, ..., I_n)) \equiv \mathbb{E}_{I_1,...,I_n \sim f_\theta^n} \left(\hat{\theta}(I_1, ..., I_n) - \theta\right)^2 \geq \frac{1}{\mathcal{I}_\theta(n)}$$

provided that we satisfy the constraint:

$$\mathbb{E}_{I_1,...,I_n \sim f_\theta^n} \left(\frac{d}{d\theta} \ln L(I_1, ..., I_n; \theta)\right) = 0$$

and where:

"Fisher information about θ"

$$\mathcal{I}_\theta(n) \equiv \mathbb{E}_{I_1,...,I_n \sim f_\theta^n} \left(\left(\frac{d}{d\theta} \ln L(I_1, ..., I_n; \theta)\right)^2\right)$$

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Analysis and Interpretation of the Cramér-Rao Lower-Bound in Astrometry: One-Dimensional Case

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PUBLICATIONS OF THE ASTRONOMICAL SOCIETY OF THE PACIFIC, **126**:798–810, 2014 August

Analysis of the Cramér-Rao Bound in the Joint Estimation of Astrometry and Photometry

RENE A. MENDEZ,[1,2] JORGE F. SILVA,[3] RODRIGO OROSTICA,[2] AND RODRIGO LOBOS[3]

PUBLICATIONS OF THE ASTRONOMICAL SOCIETY OF THE PACIFIC, **127**:1166–1182, 2015 November

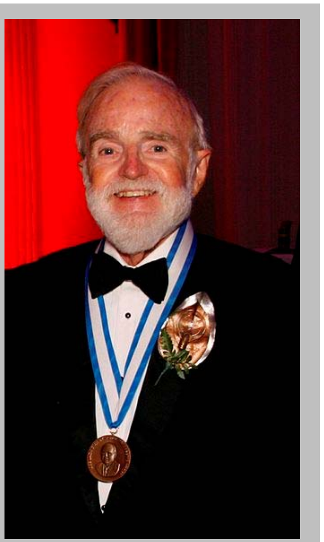
Performance Analysis of the Least-Squares Estimator in Astrometry

RODRIGO A. LOBOS,[1] JORGE F. SILVA,[1] RENE A. MENDEZ,[2] AND MARCOS ORCHARD[1]

What´s next ?

- Is the classical (parametric) Cramér-Rao lower bound the ultimate precision limit, **really** ? **No (A&A paper)**
- Can we **avoid** some of the losses in precision found on classical estimators ? **Yes (Espinosa & Lobos, in prep)**
- \+ ambitious: Can we have a **prescription** (rule/recipe) on how to do the "best" estimation of parameters ? **Yes**
- ++ ambitious: Is there an estimator that **approaches** the ultimate limit ? **Yes!**

The Bayesian CR limit



(Harry L.) van Trees (b.1930) Theorem (1968)

$$\min_{\tau^n:\mathbb{N}^n\to\mathbb{R}} \mathbb{E}_{(X_c,I^n)}\left\{(\tau^n(I^n)-X_c)^2\right\} \ge \left[\mathbb{E}_{(X_c,I^n)}\left\{\left(\frac{d\ln\tilde{L}(X_c,I^n)}{dx}\right)^2\right\}\right]^{-1}$$

But, note that, now, $\boldsymbol{X_c}$ is a *random variable*, not a *parameter*.

A *key new element* of the Bayesian approach is the prior information, which can take any mathematical form (a distribution, a bound, or any other constrain).
$\boldsymbol{\psi(x)}$

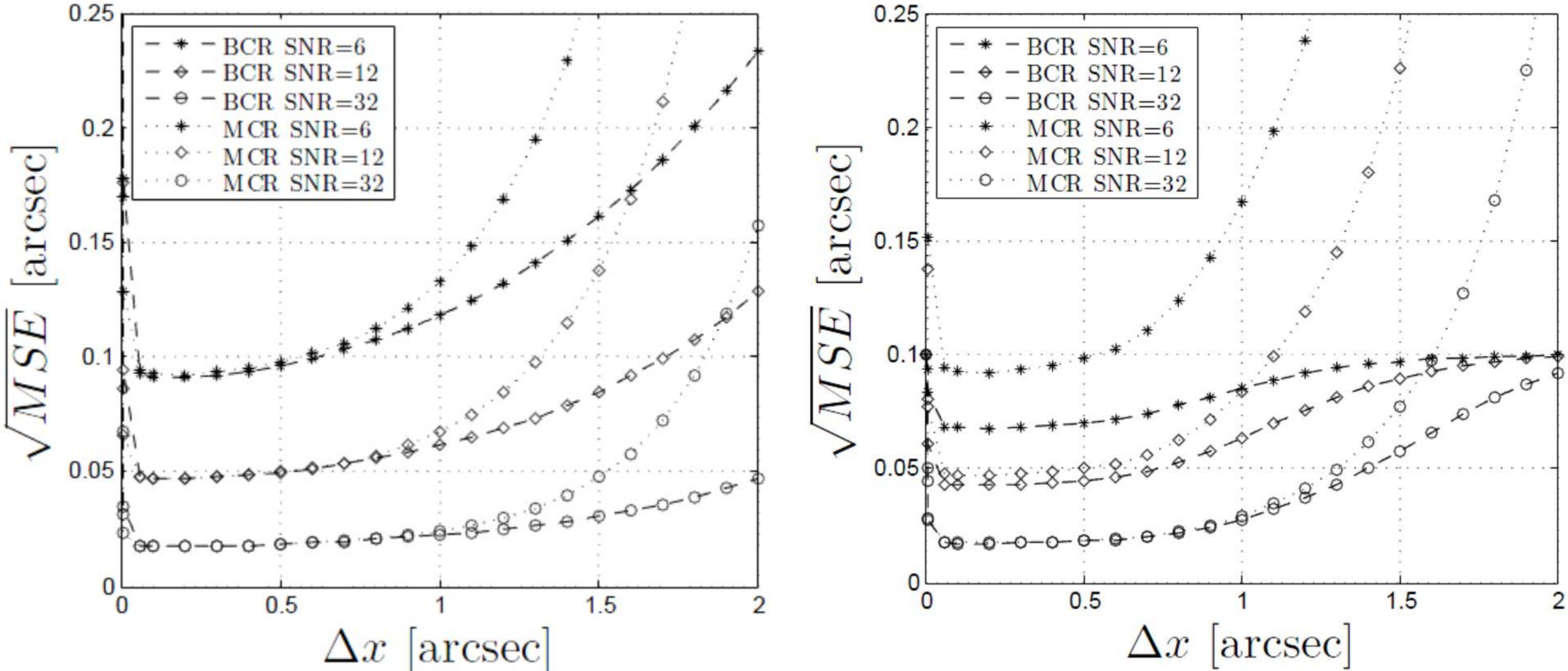
$$\sigma^2_{BCR} = \frac{1}{\mathbb{E}_{X_c \sim \psi}\{\mathcal{I}_{X_c}(n)\} + \mathcal{I}(\psi)}$$

$$\mathcal{I}(\psi) = \mathbb{E}_{X_c \sim \psi}\left\{\left(\frac{d \ln \psi(X_c)}{dx}\right)^2\right\}$$

By comparison, the classical (parametric) CR:

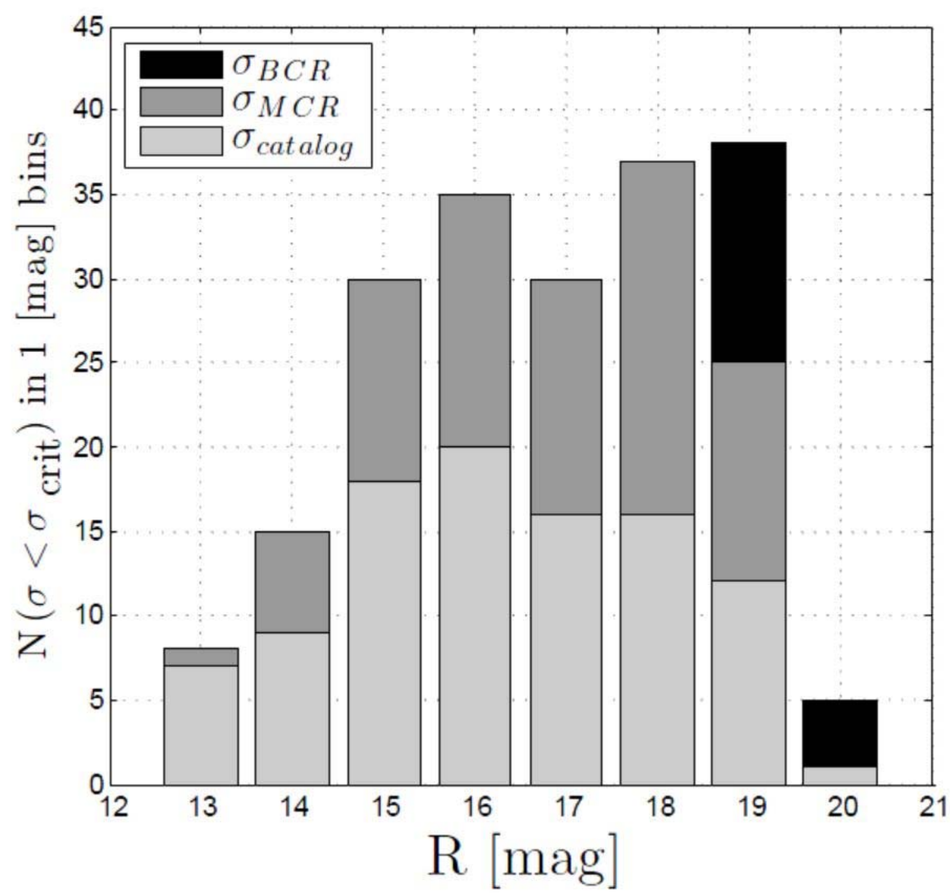
$$\sigma^2_{MCR} = \mathbb{E}_{X_c \sim \psi}\left\{\frac{1}{\mathcal{I}_{X_c}(n)}\right\}$$

➡ $\sigma^2_{BCR} \leq \sigma^2_{MCR}$ always…

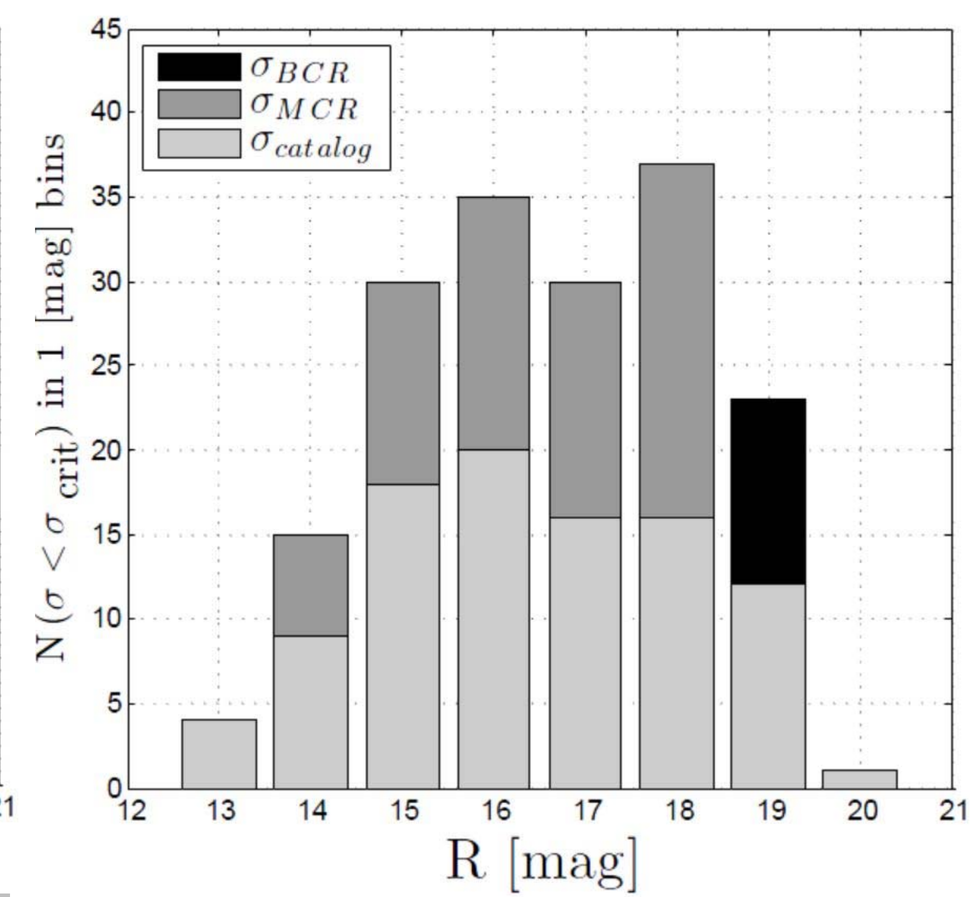

BCR SNR=6
BCR SNR=12
BCR SNR=32
MCR SNR=6
MCR SNR=12
MCR SNR=32
$\sqrt{MSE}$ [arcsec]
Δx [arcsec]
0.25
0.2
0.15
0.1
0.05
0
0.5
1
1.5
2

Example: USNO-B vs new observations @ 1 m telescope

Seeing: 1.2 arcsec



Seeing: 2.0 arcsec



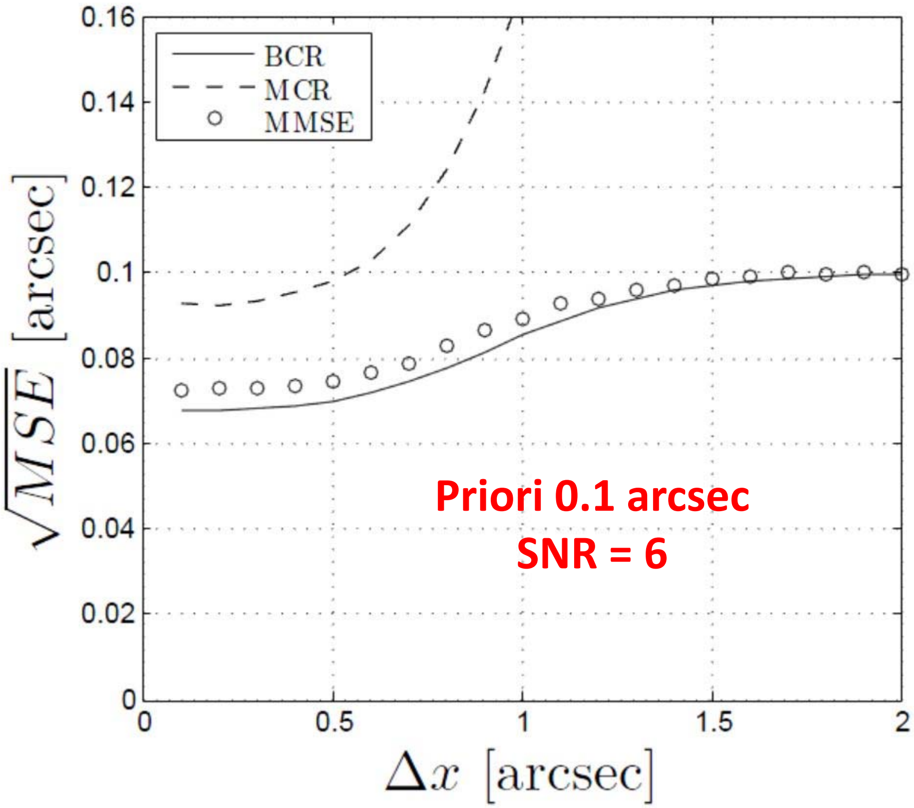
Can the gains promised by the van Trees theorem be realized in practice ?

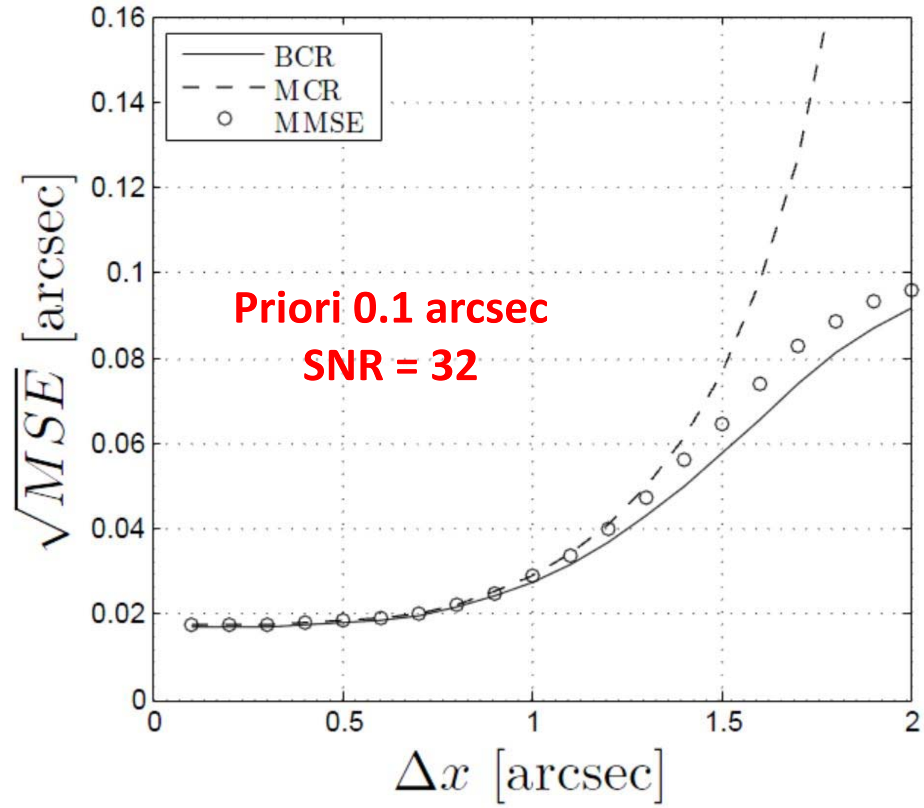
- The Minimum MSE estimator in the Bayesian context is the posterior mean for ***X_c*** (or conditional expectation, CE)
- **HUGE** advantage over parametric setting (classical CR does not provide the estimator)

- ∄ theoretical guarantee that it reaches the BCR...

$$\tau^n_{Bayes}(i^n) \equiv \mathbb{E}_{X_c|I^n=i^n}\{X_c\} = \frac{\int_{x\in\mathbb{R}} x \cdot \psi(x) p_x(i^n) dx}{\int_{\bar{x}\in\mathbb{R}} \psi(\bar{x}) p_{\bar{x}}(i^n) d\bar{x}}$$

Numerical simulations ⇒ CE (MMSE) follows the BCR **tightly** for a number of reasonable observing conditions !





The Bayesian gains can be realized using practical estimators ! ☺

Caveats & future work

- Study the BCR in photometry - uncertainty on the prior depends on the prior,
- Implement algorithms for astrometry and photometry (eventually, jointly + 2D). CE, MAP.

  Computationally very intensive, explore smart/efficient integrations

  See poster by Sebastian Espinosa

- Study the impact of miss-modelling of the prior in the BCR bound and on the eventual biases on the CE estimator

  See papers by Michalik & Lindegren on ADS: Bayesian approach to analysis of Gaia data with low SNR - or poor observational histories.

- Broader collaboration, **see poster by Ruben Clavería** on visual binary star orbits using particle filters & imputations.

To be reported on next ADeLA ! ✋

thanks for listening ☺

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