Analysis of the Bayesian Cramér-Rao Lower Bound in Photometry: Studying Achievability

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Research Problem

- Characterize the Bayesian Cramér-Rao (BCR) bound and the Mean Square Error (MSE) of the best estimator for the flux (Photometry) of point-like objects on a linear charged coupled device (CCD) detector.
- We assume access to a prior knowledge in a Bayesian setting [2] (provided by stellar catalogues) to determine the gain with respect to the classical parametric scenario.
- We also study the performance of the *Maximum a posteriori* (MAP) estimator.

Observational Setting

- Gaussian PSF

$$\phi(x,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}} [\text{arcsec}^{-1}]. \quad (1)$$

- *Background* is characterized by

$$B = f_s \Delta x + \frac{D + RON}{G} [\text{ADU}]. \quad (2)$$

- Parametric scenario: Given an unbiased estimator $\widehat{\theta}()$ and a measurement vector $\vec{I} = (I_1, ..., I_n)$ with n independent random variables driven by a poisson distribution with expectation value given by $\lambda_i(F) = G \cdot F \cdot g_i + G \cdot B$, then the Cramér-Rao bound states that:

$$\text{Var}(\widehat{\theta}(I_1, ..., I_n)) \geq \frac{1}{\mathcal{I}_{\tilde{F}}(n)}, \quad (3)$$

where $\mathcal{I}_{\tilde{F}}(n)$ is the Fisher's Information given by [1]:

$$\mathcal{I}_{\tilde{F}}(n) = \mathbb{E}\left(\left(\frac{d}{d\theta} \ln L(I_1, ..., I_n; \theta)\right)^2\right)$$

$$= \sum_{i=1}^{n} \left(\frac{\left(\frac{1}{\sqrt{2\pi}\sigma} \int_{x_k^-}^{x_k^+} e^{\gamma(x-x_c)}\right)^2}{G \cdot B + \frac{G \cdot F}{\sqrt{2\pi}\sigma} \int_{x_k^-}^{x_k^+} e^{\gamma(x-x_c)}} \right) \quad (4)$$

$\gamma(x) \equiv \frac{1}{2}(\frac{x}{\sigma})^2$, $x_i^- = x_i - \frac{\Delta x}{2}$ and $x_i^+ = x_i + \frac{\Delta x}{2}$.

References

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Bayesian Cramér-Rao Lower Bounds

If we assume that the flux F is a **random variable**; e.g., $\phi_F = N(\mu_F, \sigma_{priori})$.

$$\text{MSE}(\hat{F}) = \mathbb{E}_{I,F}[(\hat{F} - F)^2]$$

Let F be a random variable and $\vec{I} = (I_1, ..., I_n)$ a measurement vector. The Mean Square Error (MSE) of any estimator $\hat{F}$ is bounded by:

$$\mathbb{E}[(\hat{F} - F)^2] \geq \left(\mathbb{E}\left[\left(\frac{\partial \ln f_{I,F}(I,F)}{\partial F}\right)^2\right]\right)^{-1} \quad (5)$$

$$= \frac{1}{\mathbb{E}(I_F(n)) + I(\phi)} \quad (6)$$

where $\equiv \sigma^2_{BCR}$ (7)

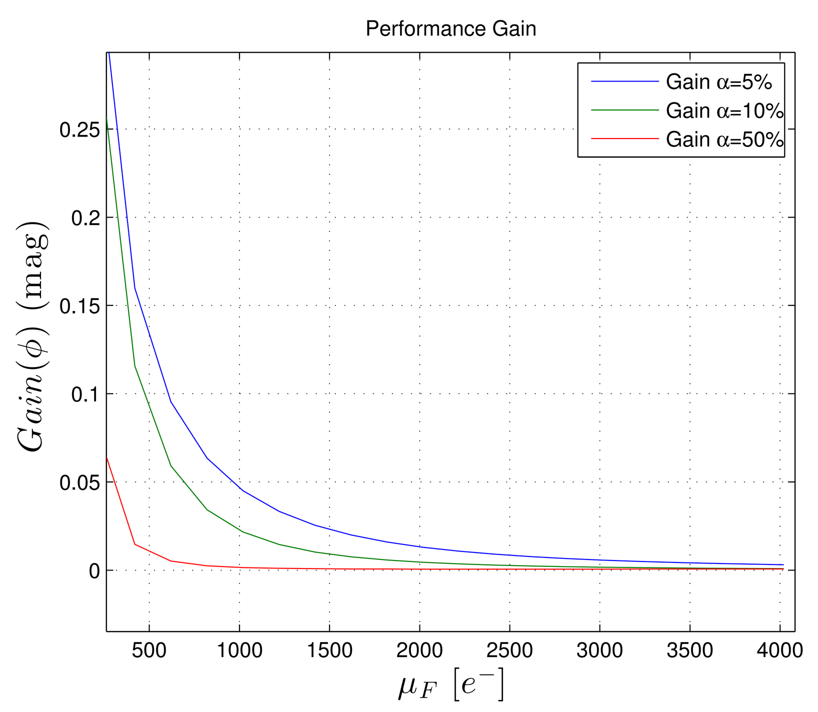
1. $I(\phi)$ is the *Prior Information*, characterized by the probability density ϕ_F.
2. $\mathbb{E}(I_F(n))$ is the *average* Fisher's Information of the parametric setting (expectation with respect to measurement data).

It is found that the BCR is always smaller than their parametric equivalents or Mean Cramér-Rao (MCR) [3], [4].

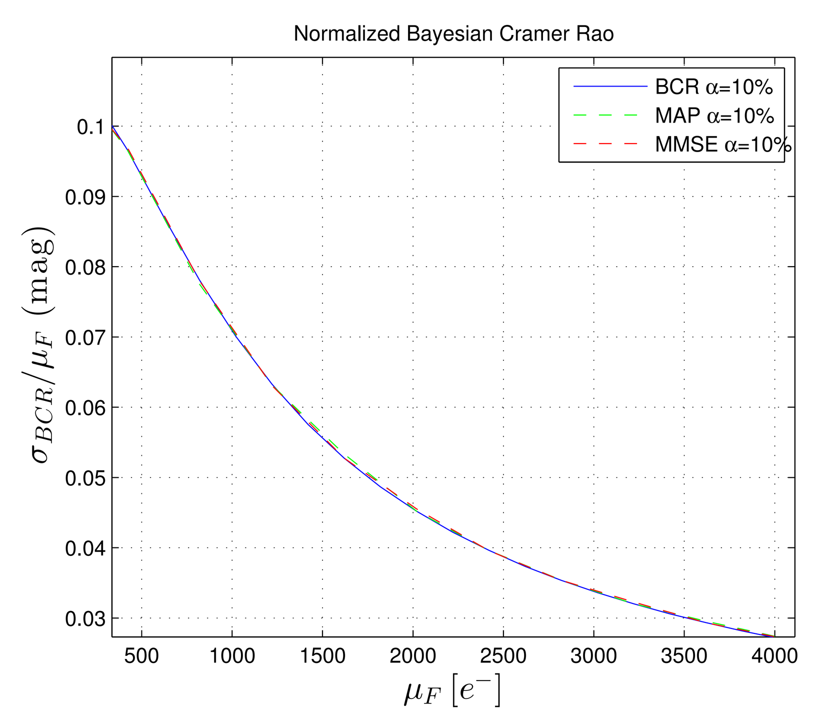
$$\frac{1}{\mathbb{E}\left(I_F(n)\right) + I(\phi)} \leq \mathbb{E}\left(\frac{1}{I_F(n)}\right) \quad (8)$$

We assume an unbiased Gaussian prior distribution $N(\mu_F, \sigma_F)$ where $\mathbb{E}(\phi) = \mu_F$.

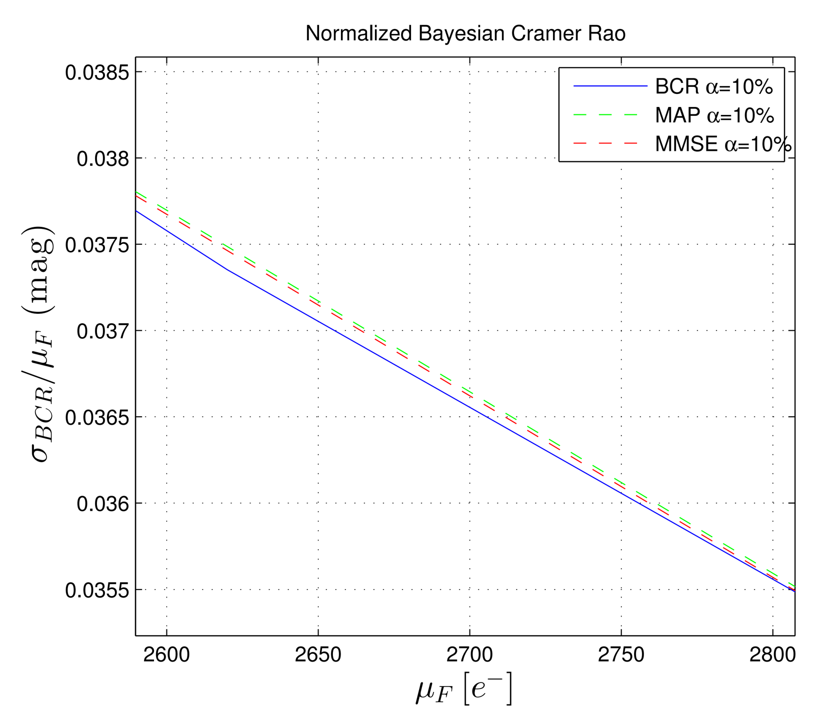
Obtained Results



Gain with respect to Mean Cramér-Rao, $B = 950$ $[e^-]$



Perfomance of Conditional Mean and MAP Rule, $B = 950$ $[e^-]$



Perfomance of Conditional Mean and MAP Rule, $B = 950$ $[e^-]$

Gain and Estimators

- We define the gain in performance for the prior as

$$\text{gain}(\phi) = \frac{\mathbb{E}\left(\frac{1}{I_F(n)}\right)}{\mathbb{E}(\phi)} - \frac{\frac{1}{\mathbb{E}(I_F(n)) + I(\phi)}}{\mathbb{E}(\phi)} \quad (9)$$

- We evaluate gain (ϕ) in different resolution scenarios (ultra high or survey precision) defined as $\sigma_F = \alpha \cdot \mu_F$, and $\alpha \in (0, 0.1]$ depending on the precision regime.
- The Minimum MSE is achievable by the posterior mean, is close to the BCR Bound (see Results).
- It can be proved that, in some regimes (e.g. with high precision) the MAP rule is an efficient estimator that reaches the BCR.

Conclusions

- The gain from the use of prior information is significant for low Signal-to-Noise regime as expected.
- In the high Signal-to-Noise regime there is no appreciable gain and, hence, BCR equals the MCR.
- In the regime of survey mode, the MAP estimator follows closely the BCR.
- Future Work: We should consider Read-Out-Noise as a random variable in the observational setting.

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